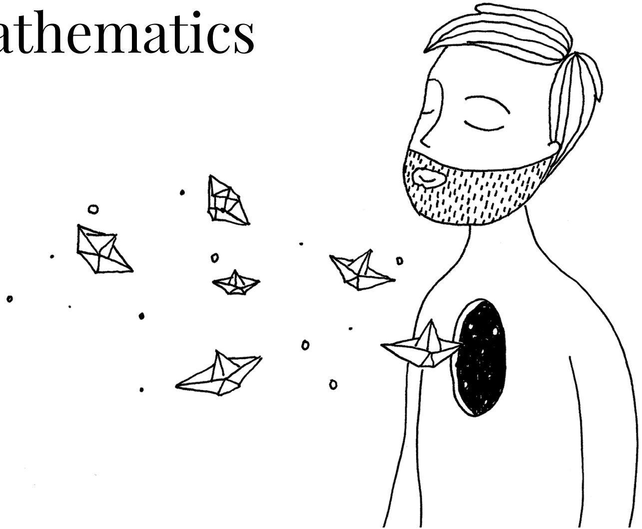
4509 - Bridging Mathematics

Topology and Continuity



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Definition

The **open ball** centered around $x_0 \in \mathbb{R}^n$ and radius $r > 0$ is defined as

$$B(x_0, r) = \{x \in \mathbb{R}^n | \quad ||x - x_0|| < r\}$$

while the **closed ball** centered around x_0 and radius $r > 0$ is

$$\overline{B}(x_0, r) = \{x \in \mathbb{R}^n | \quad ||x - x_0|| \leq r\}$$

Definition

- Let $A \subset \mathbb{R}^n$. $x_0 \in A$ is **interior**, if there is $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq A$.
- Let $A \subseteq \mathbb{R}^n$, the **interior** of A, denoted as $int(A)$, is the set of all its interior points, $int(A) = \{x \in A | \exists \epsilon > 0, B(x_0, \epsilon) \subseteq A\}$.
- The set $A \subseteq \mathbb{R}^n$ is **open** if $A \setminus int(A) = \emptyset$.
- The set A is **closed** if A^c is open.

Definition

- The **closure** of A, denoted as $\overline{A}$, is the smallest closed set that contains A.
- The **boundary** of A, denoted as ∂A, is defined as $\overline{A} \setminus int(A)$.

Definition

$A \subset \mathbb{R}^n$ is **bounded** if there is an open ball that contains A.

Definition

A set $A \subseteq \mathbb{R}^n$ is said to be **compact** if it is closed and bounded.

Definition

A **sequence** is any function $f : \mathbb{N} \rightarrow \mathbb{R}$.

Definition

The sequence x_t **converges** to x_0 if, for any open ball B containing x_0, exists $t_\epsilon \in \mathbb{N}$ such that for $t \geq t_\epsilon$, $x_t \in B$. It is denoted as $x_t \rightarrow x_0$. x_0 is called the **limit** of x_t.

Conjecture

If a sequence converges, then its limit is unique.

You know what is coming, ... Quiz! Think on a way to prove it... 10 min.

Proof.

Assume it is not unique, so:

1. $x_t \to x_0$ and also $x_t \to x_1$, and $x_1 \neq x_0$.
2. $\exists t^0_\epsilon, t^1_\epsilon \in \mathbb{N}$ such that for $t^*_\epsilon > \max\{t^0_\epsilon, t^1_\epsilon\}$ $x_t \in B(t_0, \epsilon)$ and $x_t \in B(t_1, \epsilon)$ $\forall t > t^*$.
3. Let $|x_0 - x_1| = \delta$. Choose $\epsilon = \delta/2$. So there is t^* such that $|x_t - x_0| < \delta/2$ and $|x_t - x_1| < \delta/2$.
   $|x_0 - x_1| = |x_0 - x_t + x_t - x_1| = |(x_0 - x_t) + (-x_1 + x_t)| \leq |x_0 - x_t| + |x_1 - x_t| < 2\epsilon = \delta$, contradiction!

□

Definition

- The sequence x_t is **increasing** if for any $t \in \mathbb{N}$, $x_t \leq x_{t+1} \in \mathbb{R}$.
- If x_t is increasing, it is called **bounded from above** if $x_t \leq c, \forall t \in \mathbb{N}$.

Conjecture

If the sequence x_t is increasing and bounded from above, then it converges.

Definition

Let x_t be a sequence. A **subsequence** of x_t is a sequence built by removing some of the elements of x_t without changing its order. Let $\phi : \mathbb{N} \to \mathbb{N}$ be increasing, then $y_t = x_{\phi(t)}$ is a subsequence of x_t.

Definition

Given a sequence x_t, x^* is a **cluster point** of x_t, if there is a subsequence of x_t that converges to x^*.

Conjecture

A bounded sequence converges if and only if it has only one cluster point.

Let $x_1^*, x_2^*, ..., x_p^*$ be cluster points of x_t.

Definition

- The **limit superior** (a.k.a. greatest limit, maximum limit, upper limit, lim sup, $\overline{lim}$) of x_t is defined as $\max\{x_1^*, x_2^*, ..., x_p^*\}$.
- The **limit inferior** (a.k.a. least limit, minimum limit, lower limit, lim inf, $\underline{lim}$) of x_t is defined as $\min\{x_1^*, x_2^*, ..., x_p^*\}$.

Conjecture

Let $A \subseteq \mathbb{R}^n$*.*

- *A is closed if and only if any convergent sequence* $x_t \subseteq A$ *has its limit in A. If* $x_t \subseteq A, x_t \to x_0 \Leftrightarrow x_0 \in A$*.*
- *A is compact if and only if for any sequence* $x_t \subseteq A$*, there is a convergent subsequence.*
- $\overline{A} = \{x^* | \exists x_t \in A, x_t \to x^*\}$

Definition

Let $A, C \subseteq \mathbb{R}^n$ such that $C \subseteq A$. We'll say that C is **dense** in A if and only if $\overline{C} = A$.

Definition

Consider $f : \mathbb{R}^m \to \mathbb{R}^n$. $f(x)$ **converges** to $\alpha \in \mathbb{R}^n$ when $x \in \mathbb{R}^m$ goes to $x_0 \in \mathbb{R}^m$, if for any sequence $x_n \to x_0$, $f(x_n) \to \alpha$. This is written as $\lim_{x \to x_0} f(x) = \alpha$.

Definition

$f : \mathbb{R}^m \to \mathbb{R}^m$ is **continuous** in $x_0 \in \mathbb{R}^m$ if, for any sequence $x_t \to x_0$ it holds that $f(x_t) \to f(x_0)$

Definition

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous for all $x_0 \in A \subseteq \mathbb{R}^m$, then it is continuous in A.

A more conventional definition of continuity is:

Definition

A function is said to be **continuous** on the set $S \subseteq \mathbb{R}^n$ if for every $a \in S$, and any $\epsilon > 0$ there exists δ such that for any $x \in S$ that satisfies $|x - a| \leq \delta$ implies $|f(x) - f(a)| \leq \epsilon$.

Conjecture

The sum, product, division or composition of continuous functions is continuous.

Conjecture

Let $A \subseteq \mathbb{R}^m$*, and given* $\mathcal{F} = \{f : A \to \mathbb{R}^m, f \text{ continuous in } A\}$*, it holds that* $\mathcal{F}$ *is a vector space.*

Conjecture

Let $K \subseteq \mathbb{R}^n$ *be compact and* $f : \mathbb{R}^n \to \mathbb{R}^m$ *a continuous function. Then* $f(K)$ *is compact.*

Proof.

Take a sequence $y_n \in f(K)$ that converges to some y (not necessarily in $f(K)$). Then, by definition $\exists x_n \in K$ such that $f(x_n) = y_n$. Because K is compact, there is a subsequence of x_n, say x_{n_j} that converges to some $x_0 \in K$. Now, by continuity of f, we have that $y = f(x_0) \in f(K)$ and $f(K)$ is closed.

Let's check if it is bounded. Assume it is not, and let z_n be a sequence in $f(K)$ such that $z_n \geq n$ for $n \in \mathbb{N}$. Again, repeating the argument we can get that there is some subsequence s_{n_j} in K, such that $f(s_{n_j}) = z_n$, and that converges to some $\hat{s} \in K$, because K is compact. However:

$$\infty = \lim_{n\to\infty} z_n \leq \lim_{j\to\infty} f(s_{n_j}) = f(\hat{s})$$

by the continuity of f, which is a contradiction (we found an upper bound for infinity!). □

Definition

Let $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R}$. The **maximum**(x_M) and the **minimum**(x_m) of f are defined as:

- $f(x_M) \geq f(x) \quad \forall x \in K$
- $f(x_m) \leq f(x) \quad \forall x \in K$

These are also known as *global maximum* and *global minimum*

Conjecture

Let $f : K \to \mathbb{R}$ be continuous and K compact, then x_M and x_m exist.

Definition

A set A is said to be connected if, for any $a, b \in A$, there is a continuous function $\phi : [0, 1] \to A$, such that $\phi(0) = a$ and $\phi(1) = b$.

Theorem (Bolzano)

Let $f : \mathbb{R} \to \mathbb{R}$ continuous. Let $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$, then there is $c \in \mathbb{R}$ such that $f(c) = 0$.

Theorem (Weierstrass)

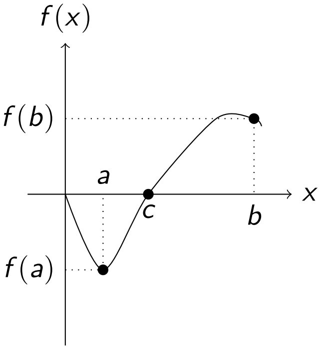
Let $[a, b] \subseteq \mathbb{R}$, let $f : [a, b] \to \mathbb{R}$ continuous. Then for any $u \in (a, b)$, there is at least one c such that $f(c) = u$.

Bolzano's.

We start with interval $I_0 = (a_0 = a, b_0 = b)$. Define $d = \frac{b+a}{2}$. There are only three possibilities:

1. $f(d) = 0$ and therefore the proof is complete, and $c = d$.
2. $f(d) < 0$, and we define interval $I_1 = (a_1 = d, b_1 = b_0)$
3. $f(d) > 0$, and we define interval $I_1 = (a_1 = a_0, b_1 = d)$

Note that $I_1 \subset I_0$, with half the length. Repeat and build a sequence of open intervals, where $I_n \subset I_{n+1}$ with $f(a_n) < 0 < f(b_n)$. Define $c_{2n} = a_n$ and $c_{2n+1} = b_n$, you have that the sequence c_i converges by the Cauchy criterion, as for $m > n$ we have $|c_m - c_n| \leq 2^{-n/2}|I_0|$. Then $c_n \to c \in [a, b]$, and given that a_n and b_n are subsequences, they converge to the same limit.
Given f continuous, $x_n \to x \Rightarrow f(x_n) \to f(x)$. We set a such that $f(a_n) \leq 0$, but $\lim_{n\to\infty} f(a_n) = f(c) \leq 0$, and the same can be said for b_n, $\lim_{n\to\infty} f(b_n) = f(c) \geq 0$, but if $f(c) \leq 0$ and $f(c) \geq 0$ then it must be that $f(c) = 0$. $\square$



f(x)
f(b)
a
x
c
b
f(a)

Brouwer fixed point theorem in $\mathbb{R}$

Theorem

Let $f : K \to K$ continuous, with $K \subseteq \mathbb{R}$ compact and convex.[1] *Then there is $\overline{x}$ such that $f(\overline{x}) = \overline{x}$.*

Proof.

- Let $f : [0, 1] \to [0, 1]$ continuous.
- Let $g(x) = f(x) - x$.
- $g(0) = f(0) - 0 = f(0)$, but $f(0) \geq 0$, so $g(0) \geq 0$
- $g(1) = f(1) - 1$, but $f(1) \leq 1$, so $f(1) - 1 \leq 0$, or $g(1) \leq 0$.
- Then, because of the proposition we just saw, there must be $\overline{x}$ such that $g(\overline{x}) = 0$, or $f(\overline{x}) = \overline{x}$.

□

[1] A.k.a. interval.

Theorem (Brower fixed point in $\mathbb{R}^n$)

Consider $B_n \subseteq \mathbb{R}^n$ the unit open ball (an open ball of radius 1). Let $f : B_n \to B_n$ continuous. Then f has a fixed point in B_n, that is, there is $x^* \in B_n$ such that $f(x^*) = x^*$.

Definition

$f : \mathbb{R}^n \to \mathbb{R}^n$ is called **locally Lipschitz continuous** if for any $x_0 \in \mathbb{R}^n$, there is a neighborhood V_{x_0} and a constant $L > 0$ such that for any $x, y \in V_{x_0}$ it holds that

$$||f(x) - f(y)|| \leq L||x - y||$$

L is called the **Lipschitz constant**.
If L does not depend on x_0, it is called simply a **Lipschitz continuous**, and furthermore, if $L < 1$ it is called a **contraction**.

Theorem (Banach fixed point)

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, then there is a single $x^* \in \mathbb{R}^n$ such that $f(x^*) = x^*$.